ON QUASI MEDIANS OF AN INVERTED WEIBULL SAMPLE

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ABSTRACT: *The paper investigates the characteristics of sampling distributions of inverted Weibull quasi medians through their moments. A comparative study of these distributions is made with reference to parameter involved.*

Key words: Inverted Weibull order statistics; Sample median and quasi median

1. INTRODUCTION

Order statistics arises naturally in many real-life applications involving data relating to life testing studies. Let $Y_1 < Y_2 <,\ldots,< Y_n$ be the order statistics based on a random sample of size n from the inverted Weibull distribution with its cdf

$$F(x)=exp\left(-\alpha^{-1}x^{-\beta}\right), \qquad x,\alpha,\beta > 0, \qquad (1.1)$$

where α and β are its scale and shape parameters respectively.

Inverted Weibull distribution have been derived as a suitable model to describe degradation phenomena of mechanical components such as the dynamic component (pistons, crankshaft, etc.) of diesel engines [1]. A lot of work has been done on inverted Weibull distribution, for example, [2] have studied the maximum likelihood and least squares estimation of inverted Weibull distribution. In [3] they studied Bayes 2-sample prediction for inverted Weibull distribution.

In [4] he explained in his work if the item consists of many parts, and each part has the same failure time distribution, and the item falls when the weakest part fails. Similarly the inverted Weibull distribution will be suitable for modeling when there are these types of applications of mechanical or electrical components lying in the life testing experiment. In [5], they discussed medians and quasi medians in their paper and derived the efficiencies of median relative to that of sample. In [6], they investigated the sampling distribution of

$$Z = a(Y_t) + b(Y_m), \quad 1 \le t \le m \le n, \qquad (1.2)$$

for some special cases where the constants $a,b > 0$ are known real numbers and find expressions for moments of linear combination of order statistics for inverted Weibull distribution.

In this paper we focus our attention on inverted Weibull quasi medians, for $\alpha = 1,$ based on a random sample of n (odd) observations to understand their properties relative to the sample median as its substitute in situations where information on median is damaged, or distorted and so unavailable for its use. Of course, both the parameter β and the sample size n cannot ignore their influence in this regard. The case when n is even is developed similarly without proof.

2. Covariance between inverted Weibull order statistics

We shall need the following theorem in the distribution of inverted Weibull quasi median. For the inverted Weibull order statistics Y_t, Y_m

Theorem :

$E(Y_t\, Y_m) =$

$$\eta \sum_{i=0}^{m-t-1}\sum_{j=0}^{n-m}(-1)^{i+j}\binom{m-t-1}{i}\binom{n-m}{j}$$

$$(\gamma_i)^{(1/\beta)-1}(\gamma_{ij})^{(1/\beta)-1}\int_0^{\infty}\int_0^{p} u^{-1/\beta}v^{-1/\beta}e^{-(u+v)}dvdu$$

where

$$\eta = n!\left[(t-1)!(m-t-1)!(n-m)!\right]^{-1}$$

$$\gamma_i = (t+i)$$

$$\gamma_{ij} = (m-t-i+j)$$

$$p = \frac{\gamma_{ij}}{\gamma_i}u$$

Proof : From the joint density of (Y_t, Y_m) for the distribution (1.1) and expanding

$$\left[e^{-y_m^{-\beta}} - e^{-y_t^{-\beta}}\right]^{m-t-1}\left[1-e^{-y_m^{-\beta}}\right]^{n-m}$$

the required expectation $E\left(Y_t, Y_m\right)$ is expressed as

$$\beta^2 n!\left[(t-1)!(m-t-1)!(n-m)!\right]^{-1}$$

$$\sum_{i=0}^{m-t-1}\sum_{j=0}^{n-m}\left[\binom{m-t-1}{i}\binom{n-m}{j}(-1)^{i+j}\right.$$

$$\times\iint \exp\begin{pmatrix}\left\{-(t+i)(y_t)^{-\beta}\right\} \\ -\left\{-(m-t-i+j)(y_m)^{-\beta}\right\}\end{pmatrix}$$

$$\times(y_t)^{-\beta}(y_m)^{-\beta}\,dy_t dy_m$$

The double integral is over the area $\left\{(y_t, y_m): 0 < y_t < y_m < \infty\right\}.$

Employing the notations and the variables

$$u = \gamma_i y_t^{-\beta}, v = \gamma_{ij} y_m^{-\beta}$$

the above expression is transformed as

$$\iint u^{-1/\beta} u^{-1/\beta} \exp(-u-v)dudv$$

where the integration is carried over

$0 < v < (\gamma_{ij} / \gamma_i)u$

in the u , v plane. We now use $p = \frac{\gamma_{ij}}{\gamma_i} u$

We now use to obtain the result.

3. Distribution of $Z_{(v-m+1)}$ for n = 2v + 1

We determine here the distribution of the inverted Weibull quasi median $Z_{(v-m+1)}$. The joint pdf of the two inverted Weibull order statistics $\left(Y_{(m)}, Y_{(2v-m+2)}\right)$ can be expressed as

$$\phi\, \beta^2 \sum_{i=0}^{2v-2m+1} \sum_{j=0}^{n-2v+m-2} \binom{2v-2m+1}{i} \binom{n-2v+m-2}{j} (-1)^{i+j} \left[\left(y_{(m)} y_{(2v-m+2)}\right)\right]^{-\beta-1}$$

$$[\exp\{-\gamma_{mi}\ (y_{(m)})^{-\beta} - \gamma_{mij}\ (y_{(2v-m+2)})^{-\beta}]$$

$0 < y_{(m)} < y_{(2v-m+2)}$

$m = 1, 2, ..., v$

$\phi = n!\left[(m-1)!(2v-2m+1)!(n-2v+m-2)!\right]^{-1}$

where

$\gamma_{mi} = m + i$

$\gamma_{mij} = 2v - 2m - i + j + 2$

To find the distribution of $Z_{(v-m+1)}$ let

$$Z = Y_{(m)} + Y_{(2v-m+2)}$$

$$W = Y_{(m)}$$

so that the pdf of (Z, W) is $h(z, w) =$

$$\phi\, \beta^2 \sum_{i=0}^{2v-2m+1} \sum_{j=0}^{n-2v+m-2} \binom{2v-2m+1}{i} \binom{n-2v+m-2}{j} (-1)^{i+j} \left[w(z-w)\right]^{-\beta-1}$$

$$[\exp\{-\gamma_{mi}\ (w)^{-\beta} - \gamma_{mij}\ (z-w)^{-\beta}]$$

$$0 < w < z < \infty \qquad (3.1)$$

The distribution of Z, and so that of $Z_{(v-m+1)}$, follows as

(3.2)

3.1 Moments of $Z_{(v-m+1)}$

We now state and prove the following theorem on the moments of $Z_{(v-m+1)}$ for the case when $(n = 2v + 1)$.

Theorem The r-th moment of inverted Weibull quasi-median $Z_{(v-m+1)}$ is given by

$$\gamma \sum_{i=0}^{2v-2m+1} \sum_{j=0}^{n-2v+m-2} \sum_{k=0}^{r} \binom{2v-2m+1}{i} \binom{n-2v+m-2}{j} \binom{r}{k} (-1)^{i+j}$$

$$\left[(\gamma_{mi})^{\frac{k}{\beta}-1} (\gamma_{mij})^{\frac{r-k}{\beta}-1} \int_0^\infty \int_0^p v_1^{-\frac{k}{\beta}} v_2^{-\frac{r-k}{\beta}} e^{-(v_1+v_2)} dv_2 dv_1 \right]$$

where

$\gamma_{mi} = m + i$,

$\gamma = (1/2)^r (n)! \left[(m-1)!(2v-2m+1)!(n-2v+m-2)!\right]^{-1}$.

$$p = \frac{\gamma_{mij}}{\gamma_{mi}} v_1 \qquad (3.1.1)$$

Proof From Eq,(3) the r-th moment of $Z_{(v-m+1)}$ is obtained from

$$(2)^{-r} \iint z^r h(z, w) dwdz.$$

which can be simplified to

$$(2)^{-r}\, \phi\, \beta^2$$

$$\sum_{i=0}^{2v-2m+1} \sum_{j=0}^{n-2v+m-2} \sum_{k=0}^{r} \binom{2v-2m+1}{i} \binom{n-2v+m-2}{j} \binom{r}{k} (-1)^{i+j}$$

$$\iint [(w)]^{k-\beta-1} [(z-w)]^{r-k-\beta-1}$$

$$[\exp\{-\gamma_{mi}\ (w)^{-\beta} - \gamma_{mij}\ (z-w)^{-\beta}]\ dw\ dz$$

$$0 < w < z < \infty \qquad (3.1.2)$$

The expression under integration comprises two terms separated by a minus sign. We consider the integral 3/kml;7

$$\iint [(w)]^{k-\beta-1} [(z-w)]^{r-k-\beta-1} \qquad (3.1.3)$$

$$[\exp\{-\gamma_{mi}\ (w)^{-\beta} - \gamma_{mij}\ (z-w)^{-\beta}]\ dw\ dz$$

On using the transformation

$$v_1 = \gamma_{mi} (w)^{-\beta}$$

$$v_2 = \gamma_{mij} (z-w)^{-\beta}$$

and the relevant Jacobian we get

$$\beta^{-2}\left(\gamma_{mi}\right)^{(k/\beta)-1}\left(\gamma_{mij}\right)^{(r-k)/\beta-1} \qquad (3.1.4)$$

$$\iint v_1^{-k/\beta} v_2^{-(r-k)/\beta}\left[\exp[(-v_1-v_2)\right]dv_2 dv_1$$

defined over (v_1,v_2) plane such that

$0<v_2<\left[\gamma_{mij}/\gamma_{mi}\right]v_1$ and now these substitutions in Eq.(3.1.2) lead to the main result .

3.2 Corollary-1: For $m=1,$ the r-th moment of the last quasi median

$Z_{(v)}=\left(Y_{(1)}+Y_{(2v+1)}\right)/2$ based on a random sample from an inverted Wibull distribution is given by

$$\gamma\sum_{i=0}^{2v-1}\sum_{j=0}^{n-2v-1}\sum_{k=0}^{r}\binom{2v-1}{i}\binom{n-2v-1}{j}\binom{r}{k}(-1)^{i+j}$$

$$\left[\left(\gamma_i\right)^{\frac{k}{\beta}-1}\left(\gamma_{ij}\right)^{\frac{r-k}{\beta}-1}\int_0^{\infty}\int_0^{p} v_1^{-\frac{k}{\beta}} v_2^{-\frac{r-k}{\beta}} e^{-(v_1+v_2)}dv_2 dv_1\right]$$

where

$\gamma_i=i+1,$

$\gamma_{ij}=2v-i+j,$

$\gamma=(1/2)^r(n)!\left[(2v-1)!(n-2v-1)!\right]^{-1}.$

$p=(\gamma_{ij}/\gamma_i)v_1$

3.2 Corollary-2: For $m=v,$ the r-th moment of the first inverted Weibull quasi median $Z_{(1)}=\left(Y_{(v)}+Y_{(v+2)}\right)/2$ is given by

$$\gamma\sum_{i=0}^{n-v-2}\sum_{j=1}^{2}\sum_{k=0}^{r}\binom{n-v-2}{i}\binom{r}{k}(-1)^{i+j}$$

$$\left[\begin{array}{l}\left(\gamma_j\right)^{\frac{k}{\beta}-1}\left(\gamma_{ij}\right)^{\frac{r-k}{\beta}-1}\\ \int_0^{\infty}\int_0^{p} v_1^{-\frac{k}{\beta}} v_2^{-\frac{r-k}{\beta}} e^{-(v_1+v_2)}dv_2 dv_1\end{array}\right]$$

Where

$\gamma_j=v-j+2$

$\gamma_{ij}=i+j$

$\gamma=(1/2)^r(n)!\left[(v-1)!(n-v-2)!\right]^{-1}$

$p=(\gamma_{ij}/\gamma_j)v_1$

4. Moments of $Z_{(v-m)}$ FOR $n=2v$

For the case when n is even the sample median and quasi medians are

$$Z_{(v)}=\left(Y_{(v)}+Y_{(v+1)}\right)/2 \qquad (4.1)$$

$$Z_{(v-m)}=\left(Y_{(v-m)}+Y_{(v-m+1)}\right)/2; \quad m=1,2,\ldots,v-1 \qquad (4.2)$$

respectively. We state here the following theorem.

4.1 Moments of $Z_{(v)}$

We now state the following theorem on the moments of $Z_{(v)}=\left(Y_{(v)}+Y_{(v+1)}\right)/2$ for the case when $(n=2v)$.

4.2 Moments of $Z_{(v-m)}$

<u>**Theorem**</u> The r-th moment of inverted Weibull quasi-median $Z_{(v-m)}$ (even) is given by

$$\gamma\sum_{i=0}^{2m}\sum_{j=0}^{n-v-m-1}\sum_{k=0}^{r}\binom{2m}{i}\binom{n-v-m-1}{j}\binom{r}{k}(-1)^{i+j}$$

$$\left[\left(\gamma_{mi}\right)^{\frac{k}{\beta}-1}\left(\gamma_{mij}\right)^{\frac{r-k}{\beta}-1}\int_0^{\infty}\int_0^{p} v_1^{-\frac{k}{\beta}} v_2^{-\frac{r-k}{\beta}} e^{-(v_1+v_2)}dv_2 dv_1\right]$$

$\gamma_{mi}=v-m+i,\ \gamma_{mij}=2m-i+j+1,$

$\gamma=(1/2)^r(n)!\times\left[(v-m-1)!(2m)!(n-v-m-1)!\right]^{-1}.$

$p=(\gamma_{mij}/\gamma_{mi})v_1$

5. Performance of first sample quasi median

In this section we compare the sample median $Y_{(v+1)}$ and the sample quasi median $\mathbf{Z_{(1)}}$ for the purpose of estimating the median Z_M of an inverted Weibull distribution, where n = 2v + 1. It is known that $Y_{(v+1)}$ has the r-th moment

$$\eta\sum_{i=0}^{n-v-1}(-1)^i\binom{n-v-1}{i}(i+v+1)^{(r/\beta)-1}$$

where $\eta=n!\left[(n-v-1)!(v)!\right]^{-1}\Gamma\left(1-(r/\beta)\right)$ (5.1)

5.1 Comparison of their expected values: We present below τ is a complex function of β, n, v and other constants. To the graphs of the median.

Z_M of an inverted Weibull distribution along with $E(Y_{(v+1)})$ and $E(\mathbf{Z}_{(1)})$ for various values of its parameter and n=3, n=25.

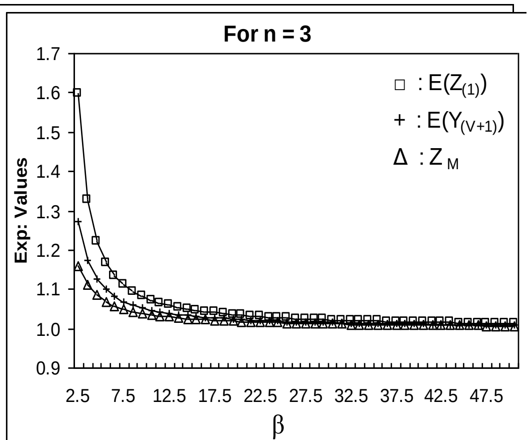

Fig 1

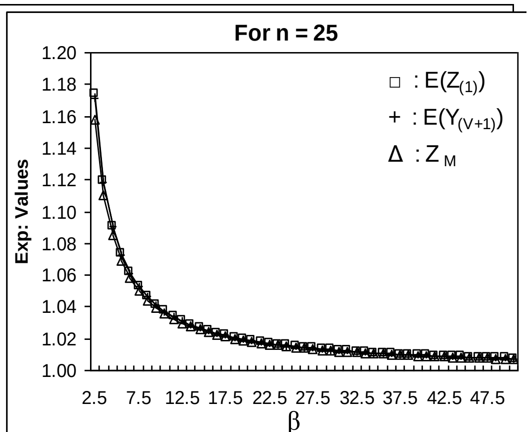

Fig 2

These graphs reveal that the sample median and quasi median overestimate the inverted Weibull median for each value of β but as its value increases the overestimation gradually fades away. However the bias due to sample median remains less than that due to quasi median. Whereas it depends on the size of a sample one essentially finds the sample median a better estimator of the population median. However the size of β also matters in this respect.

For a smaller sized sample the difference in bias for both estimators of Z_M is larger. To understand how this difference is sensitive to increasing β we consider:

$$\tau = E(Y_{(v+1)} - Z_{(1)}), \qquad (5.1.1)$$

that is, the difference between the expected value of $Y_{(v+1)}$ and $Z_{(1)}$. Both sample medians produce some bias in estimating Z_M, and this bias is related with β and the sample size n. The smaller the absolute value of τ is the larger the assurance that the quasi median can be considered as a substitute for the sample median. To study this function we calculate its numerical value and obtain the following graph for different sample sizes.

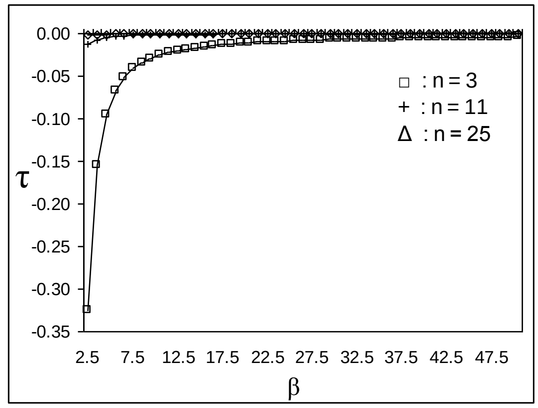

Fig 3: (β,τ)Graph

From the above graph we conclude that the quasi median overestimates the population median when sampling an inverted Weibull distribution with small sample. This overestimation gradually disappears for large samples. The difference τ also decreases as β increases for the same sample size, but it starts slightly increasing for larger β and n. For distributions with small β for small samples the inverted Weibull quasi median does not substitute well in estimating the expected value of sample median.

5.2 Mean squared errors of the two sample medians

Both sample medians provide an estimate of the inverted Weibull median. We display below the graphs in Fig-4 and Fig-5 for the mean squared errors for these sample medians for n = 3, 25.

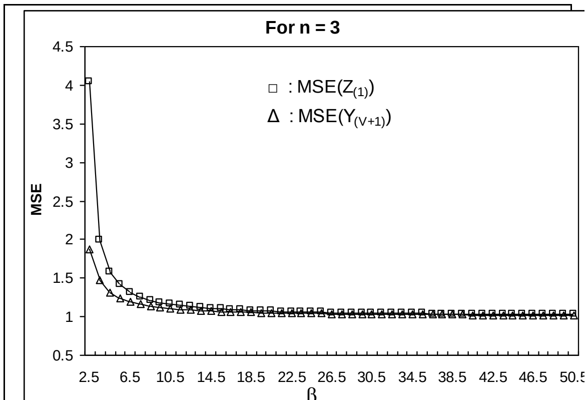

Fig 4: (β, MSE) graphs

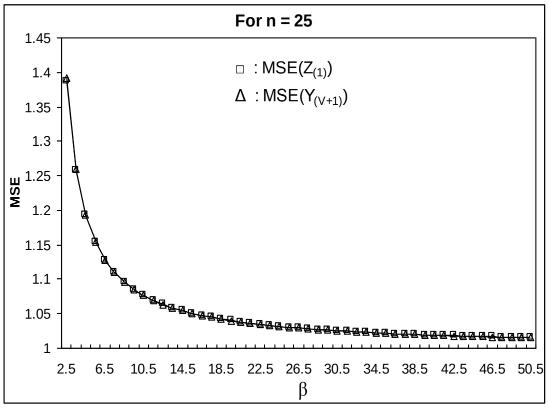

Fig 5: (β, MSE) graphs

We make the following conclusions from the above figures : The MSE ($Y_{(v+1)}$)remains less than the MSE ($Z_{(1)}$). Even with a sample size as small as 3 for $\beta \geq 10$ we find the sample quasi median $Z_{(1)}$ *a good substitute* for the sample median $Y_{(v+1)}$ in estimating the inverted Weibull median. When n, or β, increases the gap between the two MSEs decreases rapidly. To see how fast this gap between $Y_{(v+1)}$ and $Z_{(1)}$ decreases with β, we consider

$$\xi = E\,(Y_{(v+1)} - Z_{(1)})^2 \qquad (5.2.1)$$

The smaller this quantity is the less risky it is to use the quasi median as a substitute for the sample median. The quantity ξ is a complex function of β, n, v and other constants. For its evaluation, we need the terms

$$E\,(Y_{(v+1)})^2\,,\ E(Z_{(1)})^2 \text{ and } E(Y_{(v+1)}\,.\,Z_{(1)}).$$

The first quantity is found from equation(6.1). The second term follows on taking r = 2 in the above theorem. And for the third term we can write

$$2\,E(Y_{(v+1)}\,.\,Z_{(1)}) = E(Y_{(v)}\,.\,Y_{(v+1)}) + E(Y_{(v+1)}\,.\,Y_{(v+2)}) \qquad (5.2.2)$$

For the calculation of these expected values we take the joint pdf of the two indicated inverted Weibull order statistics and find the expectation of their product. This expectation takes the form of a series involving incomplete beta functions.

Alternatively, by (2.1), we can find the expectations

$$E(Y_{(v)}\,.\,Y_{(v+1)}) \text{ and } E(Y_{(v+1)}\,.\,Y_{(v+2)}) \qquad (5.2.3)$$

so that ξ in 5.2.1 can now be determined. We compute the values of ξ for various inverted Weibull distributions in the following Table:

Table 1: Values of ξ

β	n=3	n=11	n=15	n=25
2.5	1.4042	-0.0079	-0.0084	-0.0067
10.5	0.0066	-0.0003	-0.0004	-0.0003
14.5	0.0031	-0.0002	-0.0002	-0.0002
18.5	0.0018	-0.0001	-0.0001	-0.0001
22.5	0.0011	-0.0001	-0.0001	-0.0001
26.5	0.0008	-0.0001	-0.0001	0.0000
30.5	0.0006	0.0000	0.0000	0.0000
38.5	0.0004	0.0000	0.0000	0.0000
46.5	0.0002	0.0000	0.0000	0.0000

One may again discover that the expected square of the difference of two sample medians rapidly diminishes with the increase in sample size for each value of β. Also, the quantity ξ decreases rapidly when β increases for the same sample size. However, for a small sample and for β < 10 the use of inverted Weibull quasi median does not seem reasonable.

6. CONCLUDING REMARKS:

The inverted Weibull median is normally estimated by the middle observation of an ordered odd sized sample. In case this observation is damaged or lost, the above study shows that the quasi observation (near the sample median) becomes a good substitute when the sample size is not small, or when the inverted Weibull parameter is known to have a value exceeding 10.

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